

BIRATIONALITY OF BERGLUND–HÜBSCH–KRAWITZ MIRRORS

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ABSTRACT. We investigate a *multiple mirror phenomenon* arising from Berglund–Hübsh–Krawitz mirror symmetry. We prove that the different mirror Calabi–Yau orbifolds which arise in this context are in fact birational to one another.

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0. INTRODUCTION

Mirror symmetry has been a driving force in geometry and physics over the last twenty years. Roughly speaking the mirror conjecture [7] predicts a deep relationship between certain pairs (V, V°) of Calabi–Yau n -folds. At the level of cohomology for example, we have the isomorphism

$$(0.0.1) \quad H^{p,q}(V; \mathbb{C}) \cong H^{n-p,q}(V^\circ; \mathbb{C}).$$

The so called *multiple mirror phenomenon* arises when a particular Calabi–Yau V corresponds to more than one mirror V° via different mirror constructions. In many cases there is no clear relationship between the different mirrors of V which appear (see examples 0.2, 0.3 and 0.4). In this paper we investigate one such instance of the multiple mirror phenomenon coming from Berglund–Hübsh–Krawitz (BHK) mirror symmetry. We prove that the different BHK mirrors are in fact birational.

BHK mirror symmetry for invertible singularities together with the LG/CY state space correspondence of Chiodo–Ruan allows us to construct a wealth

of examples of “mirror pairs” of Calabi–Yau orbifolds arising as hypersurfaces in quotients of weighted projective space. Let (W, G) be a Landau–Ginzberg (LG) singularity. That is W is a nondegenerate quasi-homogeneous polynomial satisfying

$$W(\lambda^{c_0} X_0, \dots, \lambda^{c_n} X_n) = \lambda^d W(X_0, \dots, X_n)$$

for $\lambda \in \mathbb{C}^*$, and G is a subgroup of $\text{Aut}(W)$. Assume W satisfies the *Calabi–Yau condition*

$$\sum_{i=0}^n c_i = d.$$

Assume that G is a subgroup of $SL_{n+1}(\mathbb{C})$ and contains the distinguished element

$$j_W = \begin{pmatrix} e^{2\pi i c_0/d} \\ \vdots \\ e^{2\pi i c_n/d} \end{pmatrix},$$

acting as $j_W : (X_0, \dots, X_n) \mapsto (e^{2\pi i c_0/d} X_0, \dots, e^{2\pi i c_n/d} X_n)$. Then the group $\tilde{G} := G / \langle j_W \rangle$ acts on the projective stack $[\mathbb{P}_W] := [\mathbb{P}(c_0, \dots, c_n)]$ and we obtain a Calabi–Yau orbifold

$$[Z_W] := \{W = 0\} \subset [\mathbb{P}_W / \tilde{G}].$$

If we assume additionally that W is an invertible polynomial, we can apply BHK mirror symmetry to obtain a mirror LG singularity (W^T, G^T) , and a new Calabi–Yau

$$[Z_{W^T}] = \{W^T = 0\} \subset [\mathbb{P}_{W^T} / \tilde{G}^T].$$

This gives the mirror pair $[Z_W]$ and $[Z_{W^T}]$. By a theorem of Chiodo–Ruan [6], the Hodge diamonds of $[Z_W]$ and $[Z_{W^T}]$ are related by a 90° rotation as in (0.0.1). We say (W, G) is of *Calabi–Yau-type* if W satisfies the CY condition and $\langle j_W \rangle \leq G \leq SL_{n+1}(\mathbb{C})$.

Consider two invertible CY-type theories (W, G) and (W', G) such that the weights (c_0, \dots, c_n) of W coincide with the weights of W' and $G \leq \text{Aut}(W) \cap \text{Aut}(W')$. Then the spaces $[Z_W]$ and $[Z_{W'}]$ are both Calabi–Yau hypersurfaces in $[\mathbb{P}(c_0, \dots, c_n) / \tilde{G}]$. They are related by a smooth deformation and so give two representatives of the same mirror family. Thus the respective BHK mirrors $[Z_{W^T}]$ and $[Z_{W'^T}]$ give two different mirrors of the mirror family of $[Z_W]$. The relationship between these two orbifolds has been unknown up until now. In this paper we resolve this question.

Theorem 0.1 (= Theorem 3.1). *Let (W, G) and (W', G) be invertible and of CY-type such that the weights of W and W' are (c_0, \dots, c_n) and $G \leq \text{Aut}(W) \cap \text{Aut}(W')$ (so both $[Z_W]$ and $[Z_{W'}]$ are hypersurfaces in $[\mathbb{P}(c_0, \dots, c_n) / \tilde{G}]$). Then the respective BHK mirrors $[Z_{W^T}]$ and $[Z_{W'^T}]$ are birational.*

H. Iritani first suggested that the above might be true.

Example 0.2. Let

$$W_1 = X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5$$

and let $G = \langle j_{W_1} \rangle$. In this case $Z_{W_1} = \{W_1 = 0\} \subset \mathbb{P}^4$ is the Fermat quintic threefold in \mathbb{P}^4 . As is well known, in this case the BHK mirror coincides with the Batyrev–Borisov mirror ([2]). $W_1^T = Y_0^5 + Y_1^5 + Y_2^5 + Y_3^5 + Y_4^5$ is again a Fermat polynomial and $G_1^T = SL_5(\mathbb{C}) \cap \text{Aut}(W_1) \cong (\mathbb{Z}/5\mathbb{Z})^4$. The mirror

$$[Z_{W_1^T}] = \{Y_0^5 + Y_1^5 + Y_2^5 + Y_3^5 + Y_4^5 = 0\} \subset [\mathbb{P}^4/(\mathbb{Z}/5\mathbb{Z})^3]$$

is the well known *mirror quintic* orbifold.

Example 0.3. Consider now the degree five homogeneous polynomial of *chain type* described in Example 6 of [6],

$$W_2 = X_1^4 X_2 + X_2^4 X_3 + X_3^4 X_4 + X_4^4 X_5 + X_5^5.$$

Again let $G_2 = \langle j_{W_2} \rangle (= \langle j_{W_1} \rangle)$. From the LG model (W_2, G_2) we obtain another degree five hypersurface $Z_{W_2} = \{W_2 = 0\} \subset \mathbb{P}^4$. In this case however,

$$W_2^T = Y_1^4 + Y_1 Y_2^4 + Y_2 Y_3^4 + Y_3 Y_4^4 + Y_4 Y_5^5$$

is no longer homogeneous of degree five, but rather quasi-homogeneous:

$$W_2^T(\lambda^{64} Y_0, \lambda^{48} Y_1, \lambda^{52} Y_2, \lambda^{51} Y_3, \lambda^{41} Y_4) = \lambda^{256} W_2^T(Y_0, \dots, Y_4).$$

One can check that in this case $G_2^T = \langle j_{W_2^T} \rangle$, thus we obtain a hypersurface in weighted projective space

$$[Z_{W_2^T}] = \{W_2^T = 0\} \subset [\mathbb{P}(64, 48, 52, 51, 41)].$$

Example 0.4. Next consider the polynomial of mixed type

$$W_3 = X_1^4 X_2 + X_2^5 + X_3^5 + X_4^5 + X_5^5.$$

Again letting $G_3 = \langle j_{W_3} \rangle (= \langle j_{W_1} \rangle)$ we obtain a degree five hypersurface $Z_{W_3} = \{W_3 = 0\} \subset \mathbb{P}^4$. In this case,

$$W_3^T = Y_1^4 + Y_1 Y_2^5 + Y_3^5 + Y_4^5 + Y_5^5$$

is quasi-homogeneous:

$$W_3^T(\lambda^5 Y_0, \lambda^3 Y_1, \lambda^4 Y_2, \lambda^4 Y_3, \lambda^4 Y_4) = \lambda^{20} W_3^T(Y_0, \dots, Y_4),$$

and $G_3^T = SL_5(\mathbb{C}) \cap \text{Aut}(W_3) = \langle j_{W_3^T}, g_1, g_2 \rangle$, where

$$j_{W_3^T} = \begin{pmatrix} e^{2\pi i(1/4)} \\ e^{2\pi i(3/20)} \\ e^{2\pi i(1/5)} \\ e^{2\pi i(1/5)} \\ e^{2\pi i(1/5)} \end{pmatrix}, \quad g_1 = \begin{pmatrix} 1 \\ 1 \\ e^{2\pi i(1/5)} \\ e^{2\pi i(4/5)} \\ 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ e^{2\pi i(1/5)} \\ e^{2\pi i(4/5)} \end{pmatrix}.$$

In this case the mirror is a hypersurface in the quotient of weighted projective space

$$[Z_{W_3^T}] = \{W_3^T = 0\} \subset [\mathbb{P}(5, 3, 4, 4, 4)/(\mathbb{Z}/5\mathbb{Z})^2].$$

In the above examples, the hypersurfaces Z_{W_1} , Z_{W_2} , and Z_{W_3} are related by smooth deformations, and should therefore be considered as three different representatives of the same mirror family. However we see that there is no obvious relationship between the respective mirrors $[Z_{W_1^T}]$, $[Z_{W_2^T}]$, and $[Z_{W_3^T}]$.

0.1. Organization of the paper. In section 1 we provide a brief introduction to BHK mirror symmetry and set notation for what follows. In section 2 we relate $[Z_W]$ to $[Z_{W^T}]$ via a toric construction. In section 3 we prove the birationality of the multiple mirrors.

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1. BERGLUND–HÜBSCH–KRAWITZ MIRROR SYMMETRY

Convention 1.1. We work in the algebraic category. The term *orbifold* means “smooth separated Deligne–Mumford stack of finite type over \mathbb{C} .”

In our notation, stacks will always be written in brackets as $[X]$, and X without brackets will denote the coarse underlying space.

Berglund–Hübsh–Krawitz (BHK) mirror symmetry is defined in terms of Landau–Ginzberg theories. A Landau–Ginzberg (LG) theory is defined by a pair (W, G) where $W = W(X_0, \dots, X_n)$ is a nondegenerate quasi-homogeneous polynomial, and G is a subgroup of $\text{Aut}(W)$, the group of diagonal symmetries of W . By quasi-homogeneity we mean that there exist integers c_0, \dots, c_n such that for $\lambda \in \mathbb{C}^*$,

$$W(\lambda^{c_0} X_0, \dots, \lambda^{c_n} X_n) = \lambda^d W(X_0, \dots, X_n).$$

We assume that $\gcd(c_0, \dots, c_n) = 1$. W is then quasi-homogeneous of degree d , with *weights* c_i . We will also sometimes refer to the rational numbers $q_i = c_i/d$ as the *fractional weights* of W .

BHK mirror symmetry applies when W is a so-called *invertible* polynomial, meaning the number of monomials is equal to the number of variables. We can write

$$W = \sum_{i=0}^n \prod_{j=0}^n X_j^{e_{ij}}.$$

Let $E = (e_{ij})$ denote the exponent matrix of W . Let $E^{-1} = (e^{ij})$. Let ρ_j denote the diagonal matrix whose i^{th} diagonal entry is $\exp(2\pi i e^{ij})$. The group $\text{Aut}(W)$ is equal to $\langle \rho_0, \dots, \rho_n \rangle$. There is a distinguished element j_W whose i^{th} diagonal entry is $\exp(2\pi i q_i)$. In fact $j_W = \rho_0 \cdots \rho_n$. Given $G \leq \text{Aut}(W)$, the pair (W, G) gives an invertible LG model.

The transpose polynomial W^T is defined by transposing the exponent matrix E . Namely

$$W^T := \sum_{j=0}^n \prod_{i=0}^n \gamma_i^{e_{ij}}.$$

As above, $\text{Aut}(W^T) = \langle \bar{\rho}_0, \dots, \bar{\rho}_n \rangle$ where $\bar{\rho}_i$ is the diagonal matrix whose j^{th} diagonal entry is $\exp(2\pi i e^{ij})$. Define

$$G^T := \left\{ \prod_{i=0}^n \bar{\rho}_i^{s_i} \mid \prod_{i=0}^n X_i^{s_i} \text{ is } G\text{-invariant} \right\}.$$

The mirror LG model is given by (W^T, G^T) . See [3, 9] for more information.

We may associate to (W, G) a nondegenerate bigraded vector space, the *Fan-Jarvis-Ruan-Witten state space*

$$H_{FJRW}^*(W, G; \mathbb{C}) = \bigoplus_{p,q} H_{FJRW}^{p,q}(W, G; \mathbb{C}),$$

defined by orbifolding the classical space of Lefschetz thimbles [8]. The mirror theorem for BHK mirror symmetry then states that the Hodge diamond of $H_{FJRW}^*(W^T, G^T; \mathbb{C})$ is obtained from that of $H_{FJRW}^*(W, G; \mathbb{C})$ by performing a 90° rotation [9].

A connection between Landau–Ginzberg theories and Calabi–Yau complete intersections is given by Chiodo and Ruan in [6]. They show that for a special class of LG models (W, G) , the state space $H_{FJRW}^*(W, G; \mathbb{C})$ is isomorphic to that of a Calabi–Yau hypersurface in (a quotient of) weighted projective space. Given a polynomial

$$W = \sum_{i=0}^m \prod_{j=0}^n X_j^{e_{ij}}$$

and a group $G \leq \text{Aut}(W)$, assume that G contains the distinguished element j_W . Assume further that the weights (c_0, \dots, c_n) satisfy the *Calabi–Yau condition*

$$\sum_{i=0}^n c_i = d.$$

Let $[\mathbb{P}_W]$ denote the weighted projective stack $[\mathbb{P}(c_0, \dots, c_n)] = [(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*]$ with \mathbb{C}^* -weights (c_0, \dots, c_n) . Define $\tilde{G} := G/\langle j_W \rangle$. \tilde{G} acts on $[\mathbb{P}_W]$ and preserves the polynomial W . Thus we can define the hypersurface

$$[Z_W] := \{W = 0\} \subset [\mathbb{P}_W/\tilde{G}].$$

The theorem below relates the state space of (W, G) to the Chen-Ruan cohomology of $[Z_W]$.

Theorem 1.2 (LG/CY correspondence for hypersurfaces, [6]). *Given a Landau-Ginzberg theory (W, G) such that $\sum_{i=0}^n c_i = d$ and $j_W \in G$, there exists an isomorphism of bigraded vector spaces*

$$H_{FJRW}^{p,q}(W, G; \mathbb{C}) \cong H_{CR}^{p,q}([Z_W]; \mathbb{C}).$$

Given (W, G) as above, let us now assume that W is an invertible polynomial. One can show that $G \leq SL_{n+1}(\mathbb{C})$ if and only if G^T contains j_{W^T} , the distinguished element of W^T . Furthermore if W satisfies the Calabi–Yau condition W^T will also. So in this case (W^T, G^T) will also correspond to a hypersurface

$$[Z_{W^T}] = \{W^T = 0\} \subset [\mathbb{P}_{W^T}/\tilde{G}^T].$$

The hypersurfaces $[Z_W]$ and $[Z_{W^T}]$ give a pair of mirror Calabi–Yau orbifolds. Applying Theorem 1.2 plus BHK mirror symmetry, we see that the standard relationship between the Hodge diamonds of mirror pairs holds:

$$\begin{array}{ccc} H_{CR}^{p,q}([Z_W]; \mathbb{C}) & & H_{CR}^{n-1-p,q}([Z_{W^T}]; \mathbb{C}) \\ \downarrow \cong & & \uparrow \cong \\ H_{FJRW}^{p,q}(W, G; \mathbb{C}) & \xrightarrow{\cong} & H_{FJRW}^{n-1-p,q}(W^T, G^T; \mathbb{C}) \end{array}.$$

Definition 1.3. In this paper we deal only with Landau-Ginzberg theories (W, G) of *Calabi–Yau-type*, that is:

- i. the weights (c_0, \dots, c_n) satisfy the Calabi–Yau condition $\sum_{i=0}^n c_i = d$,
- ii. $\langle j_W \rangle \leq G \leq SL_{n+1}(\mathbb{C})$.

Convention 1.4. Henceforth, we will always assume that our polynomial W is invertible.

We will use the term *BHK mirror symmetry* to refer both to the correspondence between the pairs (W, G) and (W^T, G^T) , as well as the induced correspondence between $[Z_W]$ and $[Z_{W^T}]$.

2. REPHRASING BHK MIRROR SYMMETRY

BHK mirror symmetry reduces complex geometric relationships to a simple combinatorial construction based on transposing the exponent matrix and its inverse. Using a construction due to Clarke, we can reformulate BHK mirror symmetry, at least on the level of coarse moduli spaces, into the language of toric varieties.

Remark 2.1. In [5], Clarke gives a general construction which specializes to that detailed below.

Remark 2.2. In [4], Borisov gives a different way of rephrasing BHK mirror symmetry in toric terms.

2.1. Expressing Z_W in toric language. Let (W, G) be of CY-type. Label the i^{th} monomial of W as Y_i , so $W = \sum_{i=0}^n Y_i = \sum_{i=0}^n \prod_{j=0}^n X_j^{e_{ij}}$. We can view X_0, \dots, X_n as homogeneous coordinates on the toric variety \mathbb{P}_W/\tilde{G} , the coarse underlying space of the quotient $[\mathbb{P}_W/\tilde{G}]$. Then $\{W = 0\}$ defines the hypersurface Z_W .

Note first that \mathbb{P}_W/\tilde{G} is a normal toric variety, and can therefore be expressed as X_Σ for Σ a fan in a lattice $N \cong \mathbb{Z}^n$. Because X_Σ is a finite quotient of weighted projective space, the fan Σ takes a specific form. The following two lemmas will be useful in describing such fans and their properties.

First, it will be helpful in what follows to obtain an explicit description of the short exact sequence

$$(2.1.1) \quad 0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X_\Sigma) \rightarrow 0$$

for the divisor class group in the particular case of $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$.

Lemma 2.3. *Let (W, G) be of CY-type and let (c_0, \dots, c_n) be the weights of W . Then there exists a fan Σ such that the corresponding toric variety $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$, and the short exact sequence (2.1.1) may be written as*

$$0 \rightarrow M \rightarrow \mathbb{Z}^{n+1} \xrightarrow{\phi \oplus \tilde{\phi}} \mathbb{Z} \oplus \tilde{G} \rightarrow 0,$$

where ϕ is given by $(a_0, \dots, a_n) \mapsto (c_0 a_0 + \dots + c_n a_n)$.

Proof. We will begin by constructing a fan Σ' in a lattice $N' \cong \mathbb{Z}^n$ such that $X_{\Sigma'} \cong \mathbb{P}_W$. Define the lattice $N' := \mathbb{Z}^{n+1}/(c_0, \dots, c_n)\mathbb{Z}$ and let $M' := \text{Hom}(N', \mathbb{Z})$. Let e_j denote the standard basis in \mathbb{Z}^{n+1} and let v_j denote the image of e_j in N' . The fan Σ' for $\mathbb{P}(c_0, \dots, c_n) = \mathbb{P}_W$ consists of cones generated by proper subsets of $\{v_0, \dots, v_n\}$. Let $\rho_j := \mathbb{R}_{\geq 0} v_j$ denote the associated ray in $N'_\mathbb{R}$.

The relation $\sum_{j=0}^n c_j \cdot v_j = 0$ holds in N' , thus we have a short exact sequence

$$(2.1.2) \quad 0 \rightarrow M' \rightarrow \mathbb{Z}^{n+1} \xrightarrow{\phi} \mathbb{Z} \rightarrow 0$$

where ϕ is as above, and the map $M' \rightarrow \mathbb{Z}^{n+1}$ is given by

$$m \mapsto (\langle m, v_0 \rangle, \dots, \langle m, v_n \rangle).$$

To show that this short exact sequence is in fact the sequence (2.1.1) for the divisor class group of \mathbb{P}_W , we must prove that v_j is the generator of ρ_j for each j or, equivalently, that each v_j is primitive. This is not always the case for weighted projective space, but it does in fact hold for the weights which arise in our context.

It is a simple exercise to show that v_j can be written as $k \cdot v$ for some $v \in N'$ if and only if $k \mid \gcd(c_0, \dots, \hat{c}_j, \dots, c_n)$. Now, assume to the contrary that for \mathbb{P}_W there exists a j such that v_j is not primitive. Let $k = \gcd(c_0, \dots, \hat{c}_j, \dots, c_n)$. Then by the above $k > 1$. By assumption $\gcd(c_0, \dots, c_n) = 1$, so k does not divide c_j , and therefore does not divide $d = \sum_{j=0}^n c_j$. But

since k divides c_i for all $i \neq j$, every degree d monomial in the homogeneous variables X_0, \dots, X_n must contain a factor of X_j . This would imply that X_j divides W , contradicting our assumption that W is nondegenerate. Therefore each v_j is the generator of ρ_j , and (2.1.2) is in fact the same sequence as (2.1.1) for the fan Σ' . This proves the lemma when \tilde{G} is trivial.

In the general case, we can construct the fan Σ by first constructing $\Sigma' \subset N'$ as above and then embedding N' in an appropriate suplattice N such that $N/N' = \tilde{G}$. Let Σ denote the image of Σ' in N , it follows from standard toric geometry arguments that $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$.

We claim that v_0, \dots, v_n are still primitive in N . Recall that points in M' correspond to rational functions on $X_{\Sigma'}$, specifically, the point $m \in M'$ gives the rational function $\prod_{j=0}^n X_j^{\langle m, v_j \rangle}$. Global sections of the anticanonical sheaf are given by rational functions in X_0, \dots, X_n with at most simple poles at $X_j = 0$ for $0 \leq j \leq n$. The Calabi–Yau condition above guarantees that for each i , $Y_i/(X_0 \cdots X_n) = \prod_{j=0}^n X_j^{e_{ij}}/(X_0 \cdots X_n)$ is a rational function on \mathbb{P}_W , and can thus be represented by a point $\mu_i \in M'$.

By assumption, $G \leq SL_{n+1}(\mathbb{C})$, thus $X_0 \cdots X_n$ is invariant under the action of G . Y_i is also G -invariant, so the rational function $Y_i/(X_0 \cdots X_n)$ descends to a function on X_Σ . Equivalently, for each i , μ_i is in $M = \text{Hom}(N, \mathbb{Z})$. Now assume that for some j , v_j can be written as a multiple $v_j = k \cdot \tilde{v}_j$ for some \tilde{v}_j in N . By the nondegeneracy of W , there exists a monomial Y_i in W with no factors of X_j . Thus $Y_i/(X_0 \cdots X_n)$ will have a factor of X_j^{-1} . Thus $k\langle \mu_i, \tilde{v}_j \rangle = \langle \mu_i, k\tilde{v}_j \rangle = \langle \mu_i, v_j \rangle = -1$. Therefore k must be ± 1 . Thus v_j is primitive in N for all j .

The above discussion implies that we can write (2.1.1) for $X_\Sigma = \mathbb{P}_W/\tilde{G}$ as

$$0 \rightarrow M \rightarrow \mathbb{Z}^{n+1} \xrightarrow{\phi \oplus \tilde{\phi}} \mathbb{Z} \oplus H \rightarrow 0,$$

where ϕ is given by $(a_0, \dots, a_n) \mapsto (c_0 a_0 + \cdots + c_n a_n)$ and H is a finite group. We can identify M' with the kernel of ϕ , which results in the exact sequence

$$0 \rightarrow M \rightarrow M' \xrightarrow{\tilde{\phi}} H \rightarrow 0.$$

Applying $\text{Hom}(-, \mathbb{C}^*)$ to the above sequence, we deduce $H \cong \ker(T_{N'} \rightarrow T_N) \cong N/N' = \tilde{G}$. □

As a partial converse, we can explicitly describe the structure of X_Σ as a quotient of weighted projective space when Σ takes a certain specific form.

Lemma 2.4. *Let v_0, \dots, v_n be primitive elements of a lattice N . Assume that the sequence*

$$0 \rightarrow M \xrightarrow{A} \mathbb{Z}^{n+1} \xrightarrow{\phi \oplus \tilde{\phi}} \mathbb{Z} \oplus \tilde{G} \rightarrow 0,$$

is exact, where $A : m \mapsto (\langle m, v_0 \rangle, \dots, \langle m, v_n \rangle)$ and $\phi : (a_0, \dots, a_n) \mapsto (c_0 a_0 + \dots + c_n a_n)$ ($c_i > 0$ for all i). Define the fan $\Sigma \subset N$ by taking cones over proper faces of $\text{conv}(v_0, \dots, v_n)$. Then $X_\Sigma \cong \mathbb{P}(c_0, \dots, c_n)/\tilde{G}$.

Proof. First let $N' \subset N$ denote the lattice generated by $\langle v_0, \dots, v_n \rangle$. Define the fan $\Sigma' \subset N'$ by taking cones over proper faces of $\text{conv}(v_0, \dots, v_n)$. Note that the inclusion $N' \rightarrow N$ maps Σ' to Σ . Letting $M' = \text{Hom}(N'; \mathbb{Z})$, the sequence

$$0 \rightarrow M' \rightarrow \mathbb{Z}^{n+1} \xrightarrow{\phi} \mathbb{Z} \rightarrow 0$$

is exact, where ϕ is as above. Thus $X_{\Sigma'}$ is isomorphic to $\mathbb{P}(c_0, \dots, c_n)$. By standard toric arguments, $X_\Sigma \cong X_{\Sigma'}/(N/N')$. As in the proof of the previous lemma, the short exact sequence

$$0 \rightarrow M \rightarrow M' \xrightarrow{\tilde{\phi}} \tilde{G} \rightarrow 0$$

is exact, and after applying $\text{Hom}(-, \mathbb{C}^*)$ to the above sequence, we deduce $N/N' \cong \ker(T_{N'} \rightarrow T_N) \cong \tilde{G}$. \square

Assume we have a hypersurface $Z_W \subset \mathbb{P}_W/\tilde{G}$ corresponding to (W, G) . Let $\Sigma \subset N$ be the fan constructed in lemma 2.3 such that $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$. Let $M = \text{Hom}(N; \mathbb{Z})$ denote the dual lattice. Let $\{v_0, \dots, v_n\} \subset N$ denote the primitive elements corresponding to the generators of $\Sigma(1)$. Label them such that X_j is the homogeneous coordinate corresponding to v_j . As in the proof of lemma 2.3, each monomial Y_i in W corresponds to a rational function $Y_i/X_0 \cdots X_n$ on X_Σ , and therefore to a point $\mu_i \in M$. The hypersurface Z_W is defined by the vanishing of the section $\sum_{i=0}^n \mu_i$ of the anticanonical sheaf.

We can thus express the hypersurface $Z_W = \{W = 0\} \subset \mathbb{P}_W/\tilde{G}$ in toric language, as a toric variety $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$ plus $n+1$ points $\mu_0, \dots, \mu_n \in M$ such that $\langle \mu_i, v_j \rangle + 1 = e_{ij}$. We assemble this information in the form of a proposition as it is crucial to what follows.

Proposition 2.5. *Let (W, G) be of CY-type with $[Z_W] \subset [\mathbb{P}_W/\tilde{G}]$ the associated hypersurface. There exists a fan $\Sigma \subset N$ plus $n+1$ points $\mu_0, \dots, \mu_n \in M$ such that $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$, and Z_W is defined by the vanishing of the section $\sum_{i=0}^n \mu_i$ of $-K_\Sigma$. Thus the coarse space Z_W is determined completely by the toric variety X_Σ and the choice of sections $\mu_0, \dots, \mu_n \in M$.*

Furthermore, the construction of the fan $\Sigma \subset N$ as well as the lattice M depend only on the weights of W and the choice of group G .

Remark 2.6. We will denote the space Z_W as constructed above by

$$\left\{ \sum_{i=0}^n \mu_i = 0 \right\} \subseteq X_\Sigma$$

with the implicit assumption that the rational function $\sum_{i=0}^n \mu_i$ is viewed as a section of $-K_\Sigma$.

2.2. The BHK mirror construction using toric varieties. Applying BHK mirror symmetry to (W, G) yields the pair (W^T, G^T) , which, again applying Theorem 1.2, corresponds to a hypersurface $Z_{W^T} = \{W^T = 0\} \subset \mathbb{P}_{W^T}/\tilde{G}^T$. In fact we can relate Z_W to Z_{W^T} via a toric construction from [5] reminiscent of Batyrev's original mirror construction [1]. As in the above remark, let $\{\sum_{i=0}^n \mu_i = 0\} \subseteq X_\Sigma$ be the toric data associated to (W, G) . We will first construct a dual fan $\Sigma^\vee \subset M$.

The nondegeneracy of W implies that $\{\mu_0, \dots, \mu_n\} \subset M$ form the vertices of an n -dimensional simplex in M . The nondegeneracy of W^T guarantees each μ_i is primitive.

Definition 2.7. Given (W, G) of CY-type defining a hypersurface $\{\sum_{i=0}^n \mu_i = 0\} \subseteq X_\Sigma$, let $\Sigma^\vee \subset M$ denote the fan whose cones consist of the cones over proper faces of $\text{conv}(\mu_0, \dots, \mu_n)$.

By lemma 2.4, X_{Σ^\vee} is again isomorphic to a quotient of weighted projective space by a finite group.

We write the homogeneous coordinate ring of X_{Σ^\vee} as $\mathbb{C}[Y_0, \dots, Y_n]$, where Y_i is the coordinate associated to μ_i . Note that each v_j now corresponds to a rational function on X_{Σ^\vee} . Furthermore, we know $\langle \mu_i, v_j \rangle \geq -1$, so each v_j gives a section of $-K_{\Sigma^\vee}$. In fact the homogeneous function corresponding to the section $\sum_{j=0}^n v_j$ is exactly W^T :

$$(2.2.1) \quad W^T = \sum_{j=0}^n \prod_{i=0}^n Y_i^{\langle \mu_i, v_j \rangle + 1},$$

thus W^T gives a quasi-homogeneous function on X_{Σ^\vee} whose zero section defines a Calabi–Yau hypersurface. This gives a “mirror” Calabi–Yau:

$$\{W^T = 0\} \subset X_{\Sigma^\vee}.$$

Remark 2.8. Note that X_{Σ^\vee} , and therefore also our hypersurface, depend upon choices of μ_0, \dots, μ_n , which are determined by our choice of quasi-homogeneous polynomial W . In the case where $X_\Sigma = \mathbb{P}_W/\tilde{G}$ is a Gorenstein Fano variety and W is a Fermat polynomial, this construction coincides with the Batyrev mirror construction. More precisely, if Δ is a polytope in M such that Σ is the normal fan of Δ , then in the above situation Σ^\vee gives the normal fan of Δ° .

We claim that this construction agrees with the BHK mirror construction. Namely we will show that $X_{\Sigma^\vee} \cong \mathbb{P}_{W^T}/\tilde{G}^T$, and thus the hypersurface Z_{W^T} coincides with $\{\sum_{j=0}^n v_j = 0\} \subseteq X_{\Sigma^\vee}$.

Proposition 2.9. *Given (W, G) of CY-type, construct Σ as in Lemma 2.3 such that $X_\Sigma \cong \mathbb{P}_W/\tilde{G}$, and define Σ^\vee as in definition 2.7. Then $X_{\Sigma^\vee} \cong \mathbb{P}_{W^T}/\tilde{G}^T$.*

Proof. Recall that $\{v_0, \dots, v_n\} = \Sigma(1)$ and $\{\mu_0, \dots, \mu_n\} \subset M$ denote the rational functions corresponding to the monomials $\{Y_0, \dots, Y_n\}$. We have

the following short exact sequences:

$$(2.2.2) \quad 0 \rightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \rightarrow A_{n-1}(X_{\Sigma}) \rightarrow 0,$$

$$(2.2.3) \quad 0 \rightarrow N \xrightarrow{B} \mathbb{Z}^{\Sigma^{\vee}(1)} \rightarrow A_{n-1}(X_{\Sigma^{\vee}}) \rightarrow 0.$$

By lemma 2.3 $A_{n-1}(X_{\Sigma}) \cong \mathbb{Z} \oplus \tilde{G}$ and we can rewrite (2.2.2) as

$$(2.2.4) \quad 0 \rightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\phi \oplus \tilde{\phi}} \mathbb{Z} \oplus \tilde{G} \rightarrow 0,$$

where $\phi(a_0, \dots, a_n) = dq_0a_0 + \dots + dq_na_n$.

Note that BA^T gives the pairing matrix between $\{v_0, \dots, v_n\}$ and $\{\mu_0, \dots, \mu_n\}$,

$$(BA^T)_{ij} = \langle \mu_i, v_j \rangle.$$

Let $\mathbb{1}$ denote the $(n+1) \times (n+1)$ matrix with all entries equal to one. Then the exponent matrix E for W can be expressed as $E = BA^T + \mathbb{1}$. Now, the (fractional) weights of W^T are given by

$$\vec{p} = \begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix} = (E^T)^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Recall that $\mathbb{P}_{W^T} = \mathbb{P}(\bar{d}p_1, \dots, \bar{d}p_n)$, where \bar{d} is the smallest integer such that $\bar{d}p_i$ is an integer for all i and $\gcd(\bar{d}p_1, \dots, \bar{d}p_n) = 1$. We will show first that $X_{\Sigma^{\vee}}$ is a quotient of $\mathbb{P}(\bar{d}p_1, \dots, \bar{d}p_n)$. To see this, consider the map $\psi : \mathbb{Z}^{\Sigma^{\vee}} \rightarrow \mathbb{Z}$ given by $\psi(b_0, \dots, b_n) = \bar{d}p_0b_0 + \dots + \bar{d}p_nb_n$. We claim that $B(N) \subseteq \ker(\psi)$. This can be rephrased as saying that $\bar{d}\vec{p}^T B = \vec{0}^T$, which holds if and only if $AB^T\bar{d}\vec{p} = \vec{0}$ since A has full rank. By the Calabi-Yau condition for W^T we know that $\sum_{i=0}^n p_i = 1$, so the above statement is equivalent to the condition $E^T\vec{p} = AB^T\vec{p} + \mathbb{1}\vec{p} = \vec{0} + \vec{1} = \vec{1}$. But this is immediate as $E^T\vec{p} = E^T \cdot (E^T)^{-1}\vec{1} = \vec{1}$. Because $\gcd(\bar{d}p_1, \dots, \bar{d}p_n) = 1$, $\ker(\psi)$ contains all linear relations in $B(N)$. Thus we can rewrite (2.2.3) as

$$(2.2.5) \quad 0 \rightarrow N \xrightarrow{B} \mathbb{Z}^{\Sigma^{\vee}(1)} \xrightarrow{\psi \oplus \tilde{\psi}} \mathbb{Z} \oplus H \rightarrow 0,$$

where H is a finite group. By lemma 2.4, this implies that $X_{\Sigma^{\vee}} \cong \mathbb{P}_{W^T}/H$. More precisely, the proof of lemma 2.4 shows that $X_{\Sigma^{\vee}} = \mathbb{P}_{W^T}/(M/\langle \mu_0, \dots, \mu_n \rangle)$ and we can identify H with $M/\langle \mu_0, \dots, \mu_n \rangle$.

To complete the proof of the proposition, we will now identify $M/\langle \mu_0, \dots, \mu_n \rangle$ with \tilde{G}^T . Recall that the group G^T can be defined as

$$G^T = \left\{ \prod_{i=0}^n \bar{\rho}_i^{s_i} \mid \prod_{i=0}^n X_i^{s_i} \text{ is } G\text{-invariant} \right\}.$$

Given a G -invariant monomial $\prod_{i=0}^n X_i^{s_i}$, its degree can be expressed as $d \cdot (s_0, \dots, s_n)E^{-1}\vec{1} \in d\mathbb{Z}$. By the Calabi-Yau condition, $d \cdot (1, \dots, 1)E^{-1}\vec{1} = d \sum_{i=0}^n q_i = d$, so there exists some $k \in \mathbb{Z}$ such that $d \cdot (s_0 + k, \dots, s_n +$

$k)E^{-1}\vec{1} = 0$ or in other words, such that $\prod_{i=0}^n X_i^{s_i+k}$ is degree zero. Multiplying $\prod_{i=0}^n X_i^{s_i}$ by $\prod_{i=0}^n X_i^k$ corresponds to multiplying $\prod_{i=0}^n \overline{\rho}_i^{s_i}$ by $(j^T)^k$. Therefore \tilde{G}^T is generated by elements of the form $\prod_{i=0}^n \overline{\rho}_i^{s_i}$ where $\prod_{i=0}^n X_i^{s_i}$ is degree 0.

Given an $n+1$ -tuple (s_0, \dots, s_n) such that $\prod_{i=0}^n X_i^{s_i}$ is G -invariant and $\deg(\prod_{i=0}^n X_i^{s_i}) = 0$, the group element $\prod_{i=0}^n \overline{\rho}_i^{s_i}$ corresponds to the identity in \tilde{G}^T if, after multiplying by a multiple of j_{W^T} , it acts trivially on each coordinate, i.e., if

$$(s_0, \dots, s_n)E^{-1} + k \cdot (1, \dots, 1)E^{-1} \in \mathbb{Z}^{n+1}$$

for some $k \in \mathbb{Z}$. Equivalently, (s_0, \dots, s_n) represents the trivial element if there exists a $k \in \mathbb{Z}$ such that

$$(s_0, \dots, s_n) + k(1, \dots, 1) = (a_0, \dots, a_n)E$$

for some $a_0, \dots, a_n \in \mathbb{Z}$. In this case

$$((s_0, \dots, s_n) + k(1, \dots, 1))E^{-1}\vec{1} = (a_0, \dots, a_n)\vec{1} = \sum_{i=0}^n a_i,$$

and by assumption $(s_0, \dots, s_n)E^{-1}\vec{1} = 0$, thus $k = k(1, \dots, 1)E^{-1}\vec{1} = \sum_{i=0}^n a_i$. Since $(a_0, \dots, a_n)E = (a_0, \dots, a_n)(BA^T + \mathbb{1}) = (a_0, \dots, a_n)BA^T + (\sum_{i=0}^n a_i)(1, \dots, 1)$, it follows that $\vec{s}^T = (s_0, \dots, s_n)$ corresponds to the identity in \tilde{G}^T if and only if

$$\begin{pmatrix} s_0 \\ \vdots \\ s_n \end{pmatrix} \in \text{im}(AB^T).$$

Finally, by (2.2.2) and the fact that $G \leq SL_{n+1}(\mathbb{C})$, the vectors \vec{s} corresponding to degree zero monomials on \mathbb{P}_W preserved by G are given by $\text{im}(A)$ in $\mathbb{Z}^{\Sigma(1)}$. Thus \tilde{G}^T is identified with $\text{im}(A)/\text{im}(AB^T) \cong M/\text{im}(B^T) = M/\langle \mu_0, \dots, \mu_n \rangle$.

To see that both H and \tilde{G}^T act in the same way on \mathbb{P}_{W^T} , consider an element $m \in M$ representing an element of $H \cong M/\langle \mu_0, \dots, \mu_n \rangle$. We can choose some \vec{r} in \mathbb{Q}^{n+1} such that $B^T\vec{r} = m$. Then in coordinates m acts on the Y_i coordinate as $e^{2\pi i r_i}$. Alternatively, if we view m as a representative of an element of \tilde{G}^T , then its coordinate-wise action is given by $e^{2\pi i t_i}$ where

$\vec{t} = (E^T)^{-1}Am$. Now note that

$$\begin{aligned}
\vec{t} &= (E^T)^{-1}Am \\
&= (E^T)^{-1}AB^T\vec{r} \\
&= (E^T)^{-1}AB^T\vec{r} + (E^T)^{-1}\mathbf{1}\vec{r} - (E^T)^{-1}\mathbf{1}\vec{r} \\
&= (E^T)^{-1}E^T\vec{r} - (E^T)^{-1}\left(\sum_{i=0}^n r_i\right)\vec{1} \\
&= \vec{r} - \left(\sum_{i=0}^n r_i\right)\vec{p}.
\end{aligned}$$

Thus \vec{t} and \vec{r} differ by a (rational) multiple of $\vec{d}\vec{p}$. But \mathbb{P}_{W^T} can be expressed as a quotient as $\{\mathbb{C}^{n+1} - 0\}/\mathbb{C}^*$ where the \mathbb{C}^* weights are given by $\vec{d}\vec{p}$, thus the action of \vec{t} and \vec{r} is the same on \mathbb{P}_{W^T} . \square

Corollary 2.10. *The hypersurface Z_{W^T} is isomorphic to $\{\sum_{j=0}^n v_j = 0\} \subseteq X_{\Sigma^\vee}$.*

Proof. By proposition 2.9, $\mathbb{P}_{W^T}/\tilde{G}^T \cong X_{\Sigma^\vee}$. Furthermore, by (2.2.1), the zero set of the homogeneous function W^T is exactly the vanishing locus of $\sum_{j=0}^n v_j$. \square

3. MULTIPLE MIRRORS

Theorem 3.1. *Given (W, G) and (W', G) of CY-type such that the weights of W and W' are (c_0, \dots, c_n) and $G \leq \text{Aut}(W) \cap \text{Aut}(W')$ (so both $[Z_W]$ and $[Z_{W'}]$ are hypersurfaces in $[\mathbb{P}(c_0, \dots, c_n)/\tilde{G}]$), then the respective BHK mirrors $[Z_{W^T}]$ and $[Z_{W'^T}]$ are birational.*

Proof. Note that given a CY-type (W, G) , the corresponding hypersurface $[Z_W]$ is always an effective orbifold. In fact the set of points with nontrivial orbifold structure is a Zariski-closed subset of the space itself, thus the orbifold $[Z_W]$ is always birational to the coarse space Z_W . Therefore, to prove the birationality of $[Z_{W^T}]$ and $[Z_{W'^T}]$ it suffices to work on the level of coarse spaces. The theorem will follow once we show that Z_{W^T} and $Z_{W'^T}$ are birational.

In the toric language of remark 2.6, we can represent the hypersurfaces Z_W and $Z_{W'}$ as $\{\sum_{i=0}^n \mu_i = 0\} \subseteq X_\Sigma$ and $\{\sum_{i=0}^n \mu'_i = 0\} \subseteq X_\Sigma$ respectively. Let Σ^\vee denote the dual fan from definition 2.7 with respect to (W, G) , so $\Sigma^\vee(1) = \{\mu_0, \dots, \mu_n\}$. Let Σ'^\vee denote the fan from definition 2.7 with respect to (W', G) , so $\Sigma'^\vee(1) = \{\mu'_0, \dots, \mu'_n\}$. Both fans are supported in M (M is determined only by the weights (c_0, \dots, c_n) and the choice of group

G , see Proposition 2.5). The corresponding mirrors are given by

$$(3.0.6) \quad \begin{aligned} Z_{W^T} &= \left\{ \sum_{j=0}^n v_j = 0 \right\} \subseteq X_{\Sigma^\vee} \text{ and} \\ Z_{W'^T} &= \left\{ \sum_{j=0}^n v_j = 0 \right\} \subseteq X_{\Sigma'^\vee}. \end{aligned}$$

The toric varieties X_{Σ^\vee} and $X_{\Sigma'^\vee}$ give two different compactifications of $T_M = M \otimes \mathbb{C}^*$. By (3.0.6) both Z_{W^T} and $Z_{W'^T}$ contain $\{\sum_{j=0}^n v_j = 0\} \subset T_M$ as an open subset. This proves the claim. \square

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